

Large Deviation Property of Free Energy in p -Body Sherrington-Kirkpatrick Model

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The cumulant generating function $\phi(n)$ and rate function $\Sigma(f)$ of free energy is evaluated in the p -body Sherrington-Kirkpatrick model by the replica method with the finite replica number n . From a perturbational argument, we show that the cumulant generating function is constant in the vicinity of $n = 0$. On the other hand, with the help of two analytic properties of $\phi(n)$, the behavior of $\phi(n)$ is derived again. However, this is also shown to be broken at a finite value of n , which gives a characteristic value in the rate function near the thermodynamic value of the free energy. Through the continuation of $\phi(n)$ as a function of n , we find a way to derive the 1RSB solution at least in this model, which is to fix the RS solution to be a monotonically increasing function.

KEYWORDS: p -body Sherrington-Kirkpatrick model, large deviation property, replica method

1. Introduction

The Replica Method (RM) is a tool to describe the statics of disordered systems. It has been applied mainly to the mean-field models and succeeded to provide many interesting concepts. However, one has no idea why RM itself is a correct procedure in a mathematical sense and a satisfactory answer to the problem has not yet been obtained. There are such problems as the choice of saddle point or the so-called “analytic continuation”. These problems are particularly difficult when you have to regard them as rigorous. Thus far, only the solvable models are studied¹⁾ or the sufficient conditions of the uniqueness of the analytic continuation are examined.²⁾ On the other hand, the necessary condition for the continuation has not been extensively considered yet, except for ref. 3. Thus, as a first step toward a better understanding of the RM, we discuss these problems through the p -body Sherrington-Kirkpatrick model.⁴⁾

Hereafter, we restrict our attention to the systems that have quenched disorder and do not discuss the models that do not have explicit randomness, like structural glass. For a given disordered system with a Hamiltonian H at the inverse temperature β , the formula of RM is written as

$$[\log Z] = \lim_{n \rightarrow 0} \frac{1}{n} \log[Z^n], \quad (1.1)$$

where $Z = \text{Tr} \exp(-\beta H)$ and the bracket $[\cdots]$ denotes the configurational average with respect to the randomness in the system. The moment $[Z^n]$

is evaluated relatively easily when n is an integer. Then, the problem of RM is regarded as the continuation of the cumulant generating function $\phi(n, N) := \frac{1}{n} \log[Z^n]$ from the integer n greater than 1 to the vicinity of 0. Here, N denotes the system size.

If the continuation for any real value of n is performed, all the information on the probabilistic fluctuation of $\log Z$ is obtained.⁵⁾ For instance, the variance of the free energy is derived as

$$\left. \frac{d}{dn} \left(\frac{1}{n} \log[Z^n] \right) \right|_{n=0} = [(\log Z)^2] - [\log Z]^2,$$

and higher cumulants are also derived using corresponding higher order derivatives. Note that an analytic property of $\phi(n)$ in the vicinity of $n = 0$ is significant to the probabilistic fluctuation of the free energy. In fact, this leads to the large deviation property of the free energy.

Assuming that the self-averaging property for the free energy is held, the probability distribution function of $\log Z$ in the limit $N \rightarrow \infty$ is expressed as

$$\text{Pr}(\log Z \approx Nf) \approx \exp(-N\Sigma(f)), \quad (1.2)$$

where $\Sigma(f)$ is determined using the Legendre transformation,

$$\Sigma(f) = nf - n\phi(n), \quad f = \frac{d(n\phi(n))}{dn}, \quad (1.3)$$

with $n\phi(n) := \lim_{N \rightarrow \infty} \frac{1}{N} n\phi(n, N) = \lim_{N \rightarrow \infty} \frac{1}{N} \log[Z^n]$. This is because the original large deviation property describes the probability distribution function of the statistical

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average asymptotically:

$$\Pr \left(\sum_{i=1}^M \log Z^{(i)} \approx MNf \right) \approx \exp(-MN\Sigma(f))$$

as $M, N \rightarrow \infty$,

where $Z^{(i)}$ is the i.i.d. partition function. By setting $f = \frac{1}{N} \log Z$, the formula to determine $\Sigma(f)$ is derived by the saddle point method as

$$\begin{aligned} \log[Z^n] &= \log \int df P(f) \exp[nNf] \\ &= \log \int df \exp[N \{nf - \Sigma(f)\}] \\ &\approx N \max_f (nf - \Sigma(f)). \end{aligned}$$

The large deviation property of the free energy is evaluated in ref. 6 for the two-body Sherrington-Kirkpatrick model. In the present paper, we apply the framework of ref. 6 to the case of p -body interaction and see the similarity to ref. 7, in which the number of metastable states in the p -body SK model is evaluated. They are conceptually different but we have found that the way to study the large deviation property proposed in ref. 6 is the same as that to calculate the complexity proposed in ref. 8 for some cases.

The organization of this paper is as follows. In §2, we introduce the model and formulate a calculus of the cumulant generating function by RM with the replica number n finite. In §3, following ref. 6, we perform the perturbation for the cumulant generating function with respect to n around $n = 0$ where the first-step replica symmetry breaking (1RSB) ansatz is correct. In §4, we discuss the continuation between the RS solution for large n and the 1RSB solution for small n . In §5, the rate function Σ is explicitly calculated. Finally, §6 is devoted to our conclusions.

2. Our Model and Replica Analyses

The model Hamiltonian we discuss is given as

$$H = -\frac{1}{\sqrt{N^{p-1}}} \sum_{i_1 \dots i_p} J_{i_1 \dots i_p} S_{i_1} \dots S_{i_p}, \quad (2.1)$$

where $S_i = \pm 1$ and $J_{i_1 \dots i_p}$ is an independent random variable following the Gaussian distribution $N(0, \frac{p!}{2})$. We focus on the case $p \geq 3$. By calculating $[Z^n]$, we can write $\phi(n)$ as⁹⁾

$$\begin{aligned} \phi(n) &= \frac{\beta^2}{2n} \sum_{\alpha < \beta}^n q_{\alpha\beta}^p - \frac{1}{n} \sum_{\alpha < \beta}^n q_{\alpha\beta} \hat{q}_{\alpha\beta} + \frac{\beta^2}{4} \\ &\quad + \frac{1}{n} \log \text{Tr} \exp \left[\sum_{\alpha < \beta}^n \hat{q}_{\alpha\beta} S^\alpha S^\beta \right], \quad (2.2) \end{aligned}$$

where $q_{\alpha\beta}$ and $\hat{q}_{\alpha\beta}$ are determined by the saddle point equations of ϕ . By imposing the replica sym-

metric (RS) ansatz, the solution is obtained as

$$\begin{aligned} \phi_0(n; q, \hat{q}) &= \frac{\beta^2}{4} (n-1)q^p - \frac{n-1}{2} q\hat{q} + \frac{\beta^2}{4} + \log 2 \\ &\quad - \frac{\hat{q}}{2} + \frac{1}{n} \log \int Du \cosh^n(\sqrt{\hat{q}}u) \quad (2.3) \end{aligned}$$

with the saddle point equations

$$\begin{aligned} q &= \frac{\int Du \cosh^n(\sqrt{\hat{q}}u) \tanh^2(\sqrt{\hat{q}}u)}{\int Du \cosh^n(\sqrt{\hat{q}}u)}, \quad (2.4) \\ \hat{q} &= \frac{\beta^2}{2} p q^{p-1}, \quad (2.5) \end{aligned}$$

where Du represents the normalized Gaussian integral. The RS solution is considered to be valid for a larger value of n and higher T .

On the other hand, based on the 1RSB ansatz, the 1RSB solution is given as

$$\begin{aligned} \phi_1(n; q_0, \hat{q}_0, q_1, \hat{q}_1, m) &= \frac{\beta^2}{4} \{ (n-m)q_0^p + (m-1)q_1^p \} \\ &\quad - \frac{1}{2} \{ (n-m)q_0\hat{q}_0 + (m-1)q_1\hat{q}_1 \} + \frac{\beta^2}{4} + \log 2 - \frac{\hat{q}_1}{2} \\ &\quad + \frac{1}{n} \log \int Du \left\{ \int Dv \cosh^m(\sqrt{\hat{q}_0}u + \sqrt{\hat{q}_1 - \hat{q}_0}v) \right\}^{n/m}. \quad (2.6) \end{aligned}$$

To write the saddle point equations concisely, we define some averages as

$$\begin{aligned} \langle f \rangle_1 &:= \frac{\int Dv \cosh^m \Xi f(v)}{\int Dv \cosh^m \Xi}, \\ \langle f \rangle_0 &:= \frac{\int Du \left\{ \int Dv \cosh^m \Xi \right\}^{n/m} f(u)}{\int Du \left\{ \int Dv \cosh^m \Xi \right\}^{n/m}}, \end{aligned}$$

with $\Xi = \sqrt{\hat{q}_0}u + \sqrt{\hat{q}_1 - \hat{q}_0}v$. Then, the saddle point equations can be expressed as

$$q_0 = \langle \langle \tanh \Xi \rangle_1^2 \rangle_0, \quad \hat{q}_0 = \frac{\beta^2}{2} p q_0^{p-1} \quad (2.7)$$

$$q_1 = \langle \langle \tanh^2 \Xi \rangle_1 \rangle_0, \quad \hat{q}_1 = \frac{\beta^2}{2} p q_1^{p-1} \quad (2.8)$$

$$\frac{\beta^2}{4} (p-1)(q_1^p - q_0^p)m = \langle \log \cosh \Xi \rangle_1 - \frac{1}{m} \langle \log \int Dv \cosh^m \Xi \rangle_0. \quad (2.9)$$

3. Perturbation with Respect to n

According to ref. 4, the 1RSB solution is valid for $n = 0$ when the temperature is $T_G < T < T_c$, where T_c is the paramagnetic-spin glass transition temperature and T_G is the Gardner transition temperature,⁴⁾ which corresponds to 1RSB-full RSB transition temperature. This indicates that we have this relation $[\log Z] = \lim_{n \rightarrow 0} \phi_1(n)$. Then, we assume that the 1RSB solution describes the behavior of the cumulant generating function in the vicinity of $n = 0$ and perform the perturba-

tional analysis for the saddle point equations and the cumulant generating function itself.

First, we consider the saddle point equations of $q_0(n)$ and $\hat{q}_0(n)$ (eq. (2.7)). The previous work⁴⁾ shows $q_0(0) = \hat{q}_0(0) = 0$, so we expand them as

$$\begin{aligned} q_0(n) &= a_1 n + a_2 n^2 + \dots \\ \hat{q}_0(n) &= \hat{a}_1 n + \hat{a}_2 n^2 + \dots \end{aligned} \quad (3.1)$$

and determine their coefficients from its lower order of n . Expanding r.h.s. of the former equation of eq. (2.7) by $\sqrt{\hat{q}_0}$, we see that its \hat{q}_0 dependence is expressed only as a function of $\sqrt{\hat{q}_0}u$ in the Gaussian integral. The leading term of the expansion of q_0 is found to start from the linear \hat{q}_0 term by performing the integral. Then, the coefficient is $a_1 = \alpha \hat{a}_1$ because there are no singularities with respect to the other order parameters or the saddle point equation itself. By contrast, the latter equation of eq. (2.7) yields clearly $\hat{a}_1 = 0$ because $p - 1 \geq 2$. It turns out that $a_1 = \hat{a}_1 = 0$ and this leads again to $\hat{a}_2 = a_2 = 0$ in a similar manner. Thus, we conclude inductively that $a_k = \hat{a}_k = 0$ for all k and $q_0(n) = \hat{q}_0(n) \equiv 0$.

Substituting the result for q_0 and \hat{q}_0 in the $\phi_1(n)$, we have

$$\begin{aligned} \phi_1(n; 0, 0, q_1, \hat{q}_1, m) &= \frac{\beta^2}{4}(m-1)q_1^p - \frac{1}{2}(m-1)q_1\hat{q}_1 \\ &+ \frac{\beta^2}{4} + \log 2 - \frac{\hat{q}_1}{2} + \frac{1}{m} \log \int Dv \cosh^m(\sqrt{\hat{q}_1}u) \end{aligned} \quad (3.2)$$

Note that this expression is independent of n . Because the order parameters q_1 and \hat{q}_1 are determined by minimizing ϕ_1 , they also lose the dependence on n . Consequently, we conclude that $\phi(n) = \phi(0)$ in the vicinity of $n = 0$.

Similarly, we can perform the perturbation with respect to $\tau = \frac{T_c - T}{T_c}$. In fact, when we substitute τ for n in eqs. (3.1), we can follow the above argument and conclude that $q_0(\tau, n) = \hat{q}_0(\tau, n) = 0$ again. This implies that $\phi(\tau, n) = \phi(\tau, 0)$ for any value of $n < 0$ and in the vicinity of $\tau = 0$, which is consistent with the results shown in ref. 10. In particular, this means that $\phi(n)$ for $n < 0$ is constant if τ is sufficiently small and a discontinuous phase transition for q_0 does not occur. To confirm this result, we examine the 1RSB solution for $p = 3$ and $\beta = 3.0$ numerically by the steepest descent method for the saddle point equations with respect to the order parameters. Figure 1 shows that it is certainly a constant function at least $-5 < n < 0$.

4. Connection between RS and 1RSB solutions

We have evaluated the cumulant generating function perturbatively under the 1RSB ansatz in

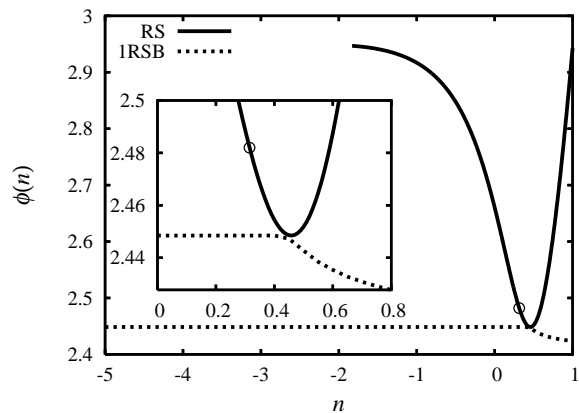


Fig. 1. n dependence of the cumulant generating function ϕ for RS (solid line) and 1RSB (dashed line) solutions for $p = 3$ and $\beta = 3.0$. The full mark represents the AT instability point n_{AT} . The inset shows an enlarged view around the minimum of the RS solutions.

the previous section. This perturbative analysis would break down at a certain value of n as n increases from 0. To discuss a breaking point, we approach it from the RS solution, which is the correct saddle point for n larger than 1 at least.

The correct cumulant generating function ϕ generally has two analytic properties, monotonicity of ϕ and convexity of $n\phi$, which can be proven using Hölder's inequality as shown in Appendix A. When one claims that an RS solution is valid at some value of n , the above two properties have to be maintained in addition to the AT stability.³⁾ Therefore, we assume that the RS solution is valid if and only if these three properties hold.

We examine whether the RS solution obtained using eq. (2.3) satisfies these properties. Let us first see the monotonicity. As shown in Fig. 1, it is clear that the monotonicity of the RS solution breaks down at some small positive value of n , which we call $n_m(\beta)$. This point is determined using the equation

$$\frac{d\phi_0}{dn} = \frac{\partial\phi_0}{\partial n} + \frac{\partial\phi_0}{\partial q} \frac{dq}{dn} + \frac{\partial\phi_0}{\partial \hat{q}} \frac{d\hat{q}}{dn} = \frac{\partial\phi_0}{\partial n} = 0. \quad (4.1)$$

The breaking point for other conditions is also determined using the equations

$$\frac{d^2\phi_0}{dn^2} = 0 \quad (4.2)$$

for the convexity and

$$1 = \frac{p(p-1)}{2} \beta^2 q^{p-2} \frac{\int Du \cosh^{n-4} \sqrt{\hat{q}} u}{\int Du \cosh^n \sqrt{\hat{q}} u} \quad (4.3)$$

for the AT stability. We define $n_{AT}(\beta)$ as the marginal solution of the AT stability condition.

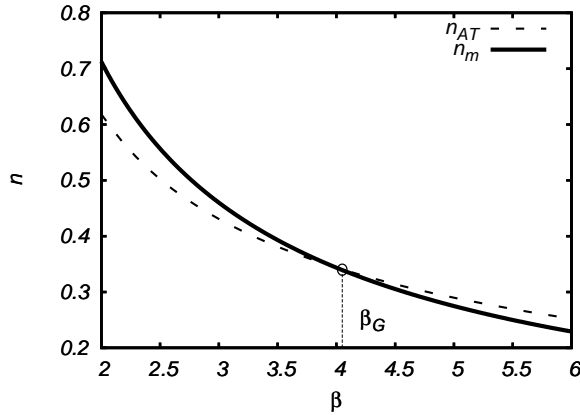


Fig. 2. $n_{AT}(\beta)$ in dashed line and $n_m(\beta)$ in solid line for $p = 3$. β_G indicates the inverse Gardner temperature.

It is shown later that the convexity is not broken when we set parameters in the 1RSB stable regime. Therefore, we consider which of the breaking points n_m and n_{AT} is relevant. As an example, we show n_m and n_{AT} for $p = 3$ as a function of the inverse temperature in Fig. 2. It is found that the two lines intersect with each other at an inverse temperature, which is denoted by β_G . We show later that this temperature β_G is the Gardner temperature. In the temperature region $\beta < \beta_G$, the monotonicity is first broken as n decreases. Thus, we conclude that $\phi(n) = \phi_0(n)$ for $n > n_m$ in the 1RSB stable temperature regime from the numerical evidence and the assumption mentioned above.

In other words, The above argument is that we take the RS solution to as small n as possible, while the 1RSB solution should be valid in the vicinity of $n = 0$. Then, we determine the cumulant generating function in the rest of the interval of n . It is, however, easily obtained by the following proposition that holds in general:

Proposition 1 *If $\phi'(n_0) = 0$ for a value of $n_0 > 0$, then $\phi(n) = \phi(0)$ for $0 < n < n_0$.*

The proof is given in Appendix A. Because we have $\phi'_0(n_m) = 0$ in eq. (4.1) by definition, this proposition yields $\phi(n) = \phi_0(0)$ for $0 < n < n_m$. It should be noted that this fact is derived without the 1RSB solution or the perturbation but with the plausible assumption.

Although the perturbation might not be necessarily required, we use it to understand intuitively what is happening to the 1RSB solution. From the perturbational argument, the order parameters turn out to be constant with respect to n . When we plot $q(x, n)$ as a function of x with the value of n changing, the shape of $q(x, n)$ is independent of n as shown in Figs. 3(a) and 3(b).

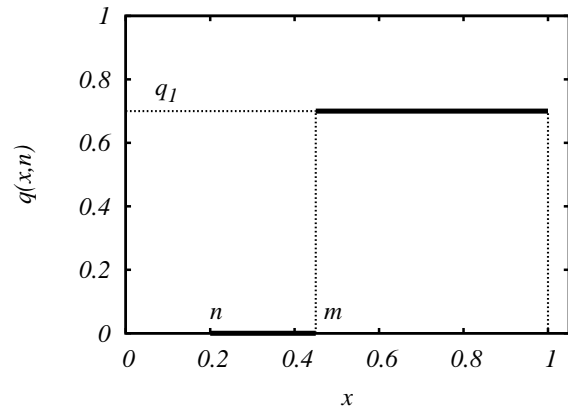


Fig. 3(a).

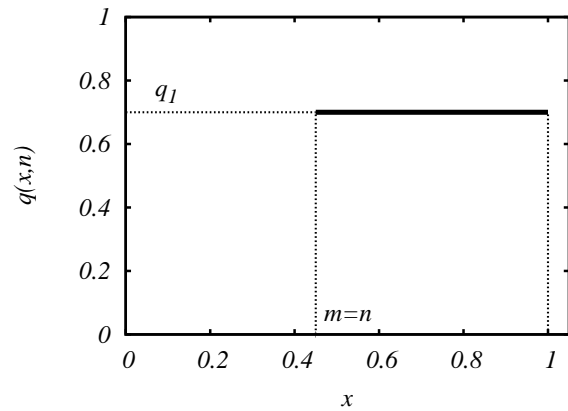


Fig. 3(b). Schematics of $q(x, n)$ for $n < m$ (a) and $n = m$ (b).

Because the lower limit of the domain of $q(x)$ is n , $q(x)$ takes only one value q_1 when $n = m$. This is how the 1RSB solution connects to the RS one as n increases from 0.

We have discussed two continuations between the RS and 1RSB solutions. One is from the RS solution to the 1RSB one with n decreasing, which indicates that the connection point is n_m . The other is from the 1RSB to the RS solutions with n increasing from 0, which indicates that it is m . Thus, this strongly suggests that $n_m = m$.

Let us compare the RS solution of eq. (2.3) to the 1RSB solution with $q_0 = 0$ of eq. (3.2). They have the same expression by identifying n with m , and q, \hat{q} with q_1, \hat{q}_1 . By definition, n_m and m give a minimal value of ϕ_0 and ϕ_1 , respectively. Because the other order parameters are also determined by minimizing respective functions, we conclude that the RS solution at $n = n_m$ coincides with the 1RSB solution including the order parameters.

This yields certainly $n_m = m$.

We now turn to the convexity of $n\phi_0$, which is written as

$$\frac{d^2\phi_0}{dn^2}(n) = \left(1, \frac{dq}{dn}, \frac{d\hat{q}}{dn}\right) \begin{pmatrix} \frac{\partial^2\phi_0}{\partial n^2} & \frac{\partial^2\phi_0}{\partial n\partial q} & \frac{\partial^2\phi_0}{\partial n\partial\hat{q}} \\ \frac{\partial^2\phi_0}{\partial n\partial q} & \frac{\partial^2\phi_0}{\partial q^2} & \frac{\partial^2\phi_0}{\partial q\partial\hat{q}} \\ \frac{\partial^2\phi_0}{\partial n\partial\hat{q}} & \frac{\partial^2\phi_0}{\partial q\partial\hat{q}} & \frac{\partial^2\phi_0}{\partial\hat{q}^2} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{dq}{dn} \\ \frac{d\hat{q}}{dn} \end{pmatrix} \geq 0. \quad (4.4)$$

From the correspondence mentioned above, the matrix in this inequality is the same as the Hessian of the 1RSB solution under the condition $q_0 = \hat{q}_0 = 0$ at $n = 0$. The matrix must be positive definite because of the stability of the 1RSB solution in the parameter region $\beta_c < \beta < \beta_G$. Then, we do not have to consider the convexity of the RS solution.

Let us next show that the temperature at which the lines $n_m(\beta)$ and $n_{AT}(\beta)$ cross is the Gardner temperature. We have defined n_{AT} as the solution of eq. (4.3) and have shown $n_m = m$. Substituting $n_m = m$ for n in eq. (4.3), we obtain that the equation is identical to the AT stability condition of the 1RSB solution.⁴⁾ Therefore, it is shown that the AT stability condition of the RS solution at $(\beta, n_m(\beta))$ is equivalent to that of the 1RSB solution at $(\beta, 0)$. This implies that the Gardner temperature is determined by the crossing temperature of $n_m(\beta)$ and $n_{AT}(\beta)$. We note that the instability of the 1RSB solution is given by the analytic properties of the RS solution at a finite value of n .

5. Calculation of Rate Function

We have the cumulant generating function $\phi(n)$ for any value of n as

$$\phi(n) = \begin{cases} \phi_0(n) & (n > n_m), \\ \phi_1(n) = \phi_0(n_m) & (n < n_m), \end{cases} \quad (5.1)$$

although an explicit formula is not given. As an example, we show in Fig. 4 the cumulant generating function evaluated numerically for $p = 3$. The function for the random energy model, corresponding to $p \rightarrow \infty$ in our model, is given rigorously as¹¹⁾

$$\lim_{p \rightarrow \infty} \phi(n) = \begin{cases} \phi_0(n) = \frac{\beta^2}{4}n + \frac{\log 2}{n} & (n > n_m), \\ \phi_1(n) = \beta\sqrt{\log 2}, & (n < n_m) \end{cases}$$

for $\beta > \beta_c$. Thus, it is confirmed that $\phi(n)$ is constant for $n < 0$ from the three independent results, perturbation with respect to τ , numerical calculations, and the exact solution for $p \rightarrow \infty$.

By means of eqs. (1.3) and (5.1), the rate function is derived in principle. Let us first evaluate

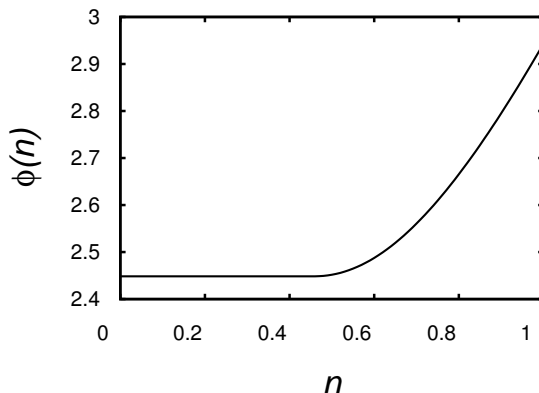


Fig. 4. Cumulant generating function for $p = 3$ and $\beta = 3.0$.

the rate function for a small fluctuation around the thermodynamic value $f_0 = \lim_{N \rightarrow \infty} \frac{1}{N} \log Z$. Noting that $\phi(n_m) = \phi(0) = f_0$ and $\phi'(n_m) = 0$, the equation $f = \frac{d(n\phi(n))}{dn}$ at $n = n_m + dn$ leads to

$$\begin{aligned} f_0 + df &= \phi(n_m + dn) + (n_m + dn)\phi'(n_m + dn) \\ &= f_0 + n_m\phi''(n_m)dn + O(dn^2), \end{aligned}$$

where df is a small and positive variable. Eventually, we have

$$dn = \frac{1}{n_m\phi''(n_m)}df, \quad (5.2)$$

and the rate function around f_0 is obtained as

$$\begin{aligned} \Sigma(f_0 + df) &= (n_m + dn)(f_0 + df - \phi(n_m + dn)) \\ &= n_m df. \end{aligned} \quad (5.3)$$

Thus, we conclude that the rate function has a linear term with respect to the fluctuation around the thermodynamic value and its coefficient is n_m . This is in contrast with the SK model with $p = 2$, where the leading term of $\Sigma(f_0 + df)$ is anomalously $(df)^{(6/5)}$ although other possibilities have also been discussed. It would be interesting to see that the finite value of the replica number n , which originates from RM, has a physical significance before taking the limit of n to 0.

The linear dependence of the rate function reminds us of a structural similarity to the thermodynamics. Rewriting the definition of $\phi(n)$ and $\Sigma(f)$, we have

$$\begin{aligned} -\frac{N}{\beta}\phi(n) &\approx -\frac{1}{\beta n} \log \left[\exp \left\{ -(\beta n) \left(-\frac{1}{\beta} \log Z \right) \right\} \right], \\ -N\Sigma(f) &\approx \log \Pr \left(\frac{1}{N} \log Z \approx f \right). \end{aligned}$$

They can be regarded as the “free energy” and the

“entropy”, respectively, when we identify the true free energy $-\frac{1}{\beta} \log Z$ as the Hamiltonian. Then, βn is an effective “temperature” in this sense and eq. (5.3) is interpreted as “ $TdS = dE$ ”, because we use f as $\log Z$ and $\Sigma(f_0) = 0$. Further comparison to thermodynamics is discussed in Appendix B.

Let us complete the calculation of $\Sigma(f)$. When $f < f_0$, $\Sigma(f) = \infty$ because $\phi(n) = \phi(0)$ for $n < 0$. When $f \gg 1$, $\Sigma(f) \approx \frac{f^2}{4\alpha}$ with α being some positive constant. This is because $\phi(n)$ is approximately proportional to n with the coefficient α when n is sufficiently large. This property is derived from these inequalities:²⁾

$$[Z^n] \geq [\exp(-n\beta H_1)] = \exp\left(\frac{\beta^2}{4} N n^2\right) \quad (5.4)$$

where H_1 is an energy value for a given configuration of $\{S_i\}$ and for $n \in \mathbf{N}$

$$\begin{aligned} [Z^n] &= \text{Tr}[\exp(-\sum_{a=1}^n \beta H^{(a)})] \\ &= \text{Tr} \exp \left\{ \frac{\beta^2 p!}{4N^{p-1}} \sum_{i_1 < \dots < i_p} \left(\sum_{a=1}^n S_{i_1}^{(a)} \dots S_{i_p}^{(a)} \right)^2 \right\} \\ &\leq 2^{Nn} \exp\left(\frac{\beta^2}{4} N n^2\right). \end{aligned} \quad (5.5)$$

Thus, for all $n' \in \mathbf{R}$, $\phi(n') \leq \phi(n+1) \leq \frac{\beta^2}{4}(n+1) + \log 2$, where n is the largest integer which does not exceed n' . These imply that $\alpha = \frac{\beta^2}{4}$.

To fix the normalization factor \mathcal{N} , we introduce a characteristic value F of the free energy at which the expression of $\Sigma(f)$ changes. For sufficiently large N , the probability distribution is expressed as (1.2) with $\Sigma(f)$. Then, the normalization condition can be approximately written as

$$\begin{aligned} 1 &= \int_{f_0}^F df \mathcal{N} e^{-N n_m (f-f_0)} + \int_F^\infty df \mathcal{N} e^{-N \alpha f^2} \\ &= \int_0^{N(F-f_0)} \frac{df}{N} \mathcal{N} e^{-n_m f} + \int_{\sqrt{N}F}^\infty \frac{df}{\sqrt{N}} \mathcal{N} e^{-\alpha f^2}. \end{aligned}$$

Because the second term is a Gaussian integral whose integral range does not include its peak, it vanishes exponentially as N increases. Then, we find that the leading term of \mathcal{N} is $N n_m$. By using this normalization factor, the mean of the fluctuation $f_0 + df$ is evaluated as

$$[df] = \frac{1}{N n_m}. \quad (5.6)$$

Thus it decays as N^{-1} and this should be confirmed numerically.

6. Summary and Conclusions

In this paper, we have calculated the cumulant generating function and the rate function of p -body Sherrington-Kirkpatrick model in the 1RSB temperature regime and argued the probabilistic property of the free energy $\log Z$.

We have evaluated the cumulant generating function by using RM with the replica number n finite from the two viewpoints. The perturbation analysis based on the 1RSB solution near $n = 0$ indicates that the free energy and the order parameters are constant as a function of n in the vicinity of $n = 0$. From the assumption that the RS solution is valid for $n > n_m$ when $\beta < \beta_G$, it follows that if we can take the RS solution down to $n = n_m$, the cumulant generating function for $0 < n < n_m$ is determined uniquely as a constant function.

Performing the Legendre transformation, we have evaluated the rate function, whose leading term near the thermodynamic value f_0 have been found to be $\Sigma(f_0 + df) = n_m df$ for $0 < df \ll 1$. This form is similar to the conventional thermodynamics if we regard the free energy as the “Hamiltonian”.

From these arguments, we have considered that the monotonicity breaking point n_m is more relevant than the AT stability breaking point n_{AT} when the temperature is in the 1RSB regime. It is because the crossing point of n_m and n_{AT} as a function of temperature is identical with the Gardner temperature at least in the model discussed. This criterion for determining the Gardner temperature is formally equivalent to that shown in ref. 12, where a finite-temperature phase diagram of a coloring problem on a finite connectivity graph is discussed. We suppose that the monotonicity breaking explains the reason why the breaking parameter m is determined “thermodynamically”.

Furthermore, this is one of the possible mechanisms of how an RS solution breaks to a 1RSB solution. Note that it is not based on the AT stability breaking, but on the monotonicity breaking. The latter case yields only the 1RSB solution in principle. This scenario might explain why the RS solution breaks down only one step in many other models. In other words, the 1RSB solution could be derived from the RS solution fixed as a monotonically increasing function. Thus, using this scheme, one may calculate the “1RSB” free energy even if the standard 1RSB Parisi matrix is hardly constructed, which is the case in such models as finite-connectivity systems.

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Appendix A: Proof of Proposition 1

In the Appendix, we prove the proposition 1 mentioned in §4. The proof relies on the following lemmas concerning the general properties of the cumulant generating function:

- (1) (monotonicity) $\phi(n)$ is a monotonically increasing function in a broad sense.
- (2) (convexity) $n\phi(n)$ is a convex function in a broad sense.

Their proofs are as follows: From the Hölder’s inequality, we have for $n < m$

$$[Z^n] = [Z^m]^{\frac{n}{m}} < [Z^m]^{\frac{n}{m}}. \quad (\text{A.1})$$

From dividing the logarithm of both sides by nN , it follows that $\phi_N(n) < \phi_N(m)$. In the limit $N \rightarrow \infty$, the monotonicity is proven.

From the Hölder’s inequality again, for $s < t$ and $0 < \alpha < 1$, we have

$$[Z^{\alpha s + (1-\alpha)t}] < [Z^s]^\alpha [Z^t]^{1-\alpha}. \quad (\text{A.2})$$

In a similar manner, this leads to

$$(\alpha s + (1-\alpha)t)\phi_N(\alpha s + (1-\alpha)t) < \alpha s\phi_N(s) + (1-\alpha)t\phi_N(t).$$

In the limit $N \rightarrow \infty$, this proves the convexity.

From the monotonicity, we have

$$\phi(n) \leq \phi(n_0) \quad (\text{A.3})$$

for $n < n_0$, while from the convexity, we have

$$n\phi(n) \geq (n_0\phi'(n_0) + \phi(n_0))(n - n_0) + n_0\phi(n_0),$$

where the right-hand side represents the tangent line at $n = n_0$. Because $\phi'(n_0) = 0$, this inequality is reduced to $n\phi(n) \geq n\phi(n_0)$. It turns out that for $n > 0$,

$$\phi(n) \geq \phi(n_0). \quad (\text{A.4})$$

The conditions (A.3) and (A.4) are satisfied simultaneously only when $\phi(n) = \phi(n_0)$. This completes the proof of the proposition.

Appendix B: Entropy Interpretation

Here, we comment on the non-negativity condition of the rate function, $\Sigma(f) \geq 0$, which is de-

rived in ref. 13 from the fact that the probability distribution function has to be normalizable. From this condition, the cumulant generating function $\phi(n)$ is restricted as

$$\frac{d(n\phi(n))}{dn} \geq \phi(n),$$

which is reduced to

$$n\phi'(n) \geq 0.$$

Thus, we see that the non-negativity of $\Sigma(f)$ corresponds to the monotonicity condition of $\phi(n)$ for $n \geq 0$.

This condition is a kind of “entropy crisis”, because Σ can be interpreted as entropy induced by the randomness. We also consider the condition imposed from the real entropy crisis $S = \beta^2 \frac{d}{d\beta} \left(-\frac{1}{\beta} \log Z \right) \geq 0$, which is

$$\frac{\partial}{\partial \beta} \psi(\beta, \gamma) \geq 0,$$

where

$$\begin{aligned} \psi(\beta, \gamma) &:= -\frac{1}{\gamma} \log[\exp(-\gamma F(\beta))] \\ &= -\frac{1}{\beta} \phi(\gamma/\beta). \end{aligned}$$

Note that the sign of $\frac{\partial \phi}{\partial \beta}$ is not determined. In this sense, this paraphrase suggests that $\gamma = \beta n$ is a more natural parameter than n .

- 1) K. Ogure and Y. Kabashima: Prog. Theor. Phys. **111** (2004) 661.
- 2) J. L. van Hemmen and R. G. Palmer: J. Phys. A **12** (1979) 563.
- 3) I. Kondor: J. Phys. A **16** (1983) L127.
- 4) E. Gardner: Nucl. Phys. B **257** (1985) 747.
- 5) A. Crisanti, G. Paladin, H.-J. Sommers, and A. Vulpiani: J. Phys. I **2** (1992) 1325.
- 6) G. Parisi and T. Rizzo: cond-mat/0706.1180v2.
- 7) A. Montanari and F. Ricci-Tersenghi: Eur. Phys. J. B **33** (2003) 339.
- 8) R. Monnason: Phys. Rev. Lett. **75** (1995) 15 2847.
- 9) H. Nishimori: *Statistical Physics of Spin Glasses and Information Processing: An Introduction* (Oxford University Press, New York, 2001)
- 10) V. Dotsenko, S. Franz and M. Mézard: J. Phys. A **27** (1994) 2351.
- 11) M. Talagrand: J. Stat. Phys. **126** (2007) 837.
- 12) F. Krzakala and L. Zdeborova: Eur. Phys. Lett. **81** (2008) 57005
- 13) M. Inoue, K. Hukushima, M. Okada, and Y. Kabashima: 2004 Workshop on Information-Based Induction Sciences (IBIS2004)[in Japanese].